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On the existence of local and global Lagrangians for ordinary differential equations

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Abstract. Necessary and sufficient conditions for the existence of local and global Lagrangians for ordinary differential equations of arbitrary order are described in terms of the geometry of higher-order tangent bundles. The results are applied to the study of gauge invariant differential equations and the second-order differential equation defined by the $(2+1)$-dimensional Yang-Mills Lagrangian with the Chern-Simons term is discussed.

1. Introduction

Necessary and sufficient conditions for the existence of local and global variational principles for systems of partial differential equations have been discussed (see for example [1-3] for the local existence and [4, 5] for the obstructions to the existence of global Lagrangian densities). At the same time, a geometrical description of the inverse problem in Lagrangian dynamics in terms of the geometry of the tangent bundle was depicted in [6] providing necessary and sufficient conditions for the existence of local Lagrangians for second order differential equations (SODE). In this paper we show that not only the geometry of the (local) inverse problem for SODEs but also the geometry of the (local) inverse problem for kth-order differential equations can be described in terms of the general inverse problem and vice versa, both approaches being related by the variational derivative, the first-order differential operator on forms associated to the exterior differential in the path space of the configuration space of the system, described extensively in [7-9] and from where we take most of our insights. It is remarkable that the condition for the existence of a local variational principle, easily expressed in the path space as the closedness of the Euler-Lagrange 1-form, becomes a rather intrincate geometrical property in terms of the intrinsic geometry of systems of differential equations in higher-order tangent bundles. At the same time we will indicate how the self-adjointness conditions for SODEs are recast in this geometrical setting.

We also discuss the obstructions to the existence of global Lagrangians. Our results are reformulations using the geometry of higher-order tangent bundles of results described in [4, 5]. We show that the main obstruction to the existence of a global

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Lagrangian is a second cohomology class in the configuration space of the system. This obstruction can be removed iff it is of integral class as has been already suggested in [10]; we provide some simple formulae to compute it. Finally we will apply these results to the simple, but interesting case of gauge-invariant systems of differential equations. We will show that the reduced system is locally Lagrangian iff the unreduced system is locally Lagrangian, but at the same time, the reduced system does not have to be globally Lagrangian even if the unreduced system has a global Lagrangian. We will compute explicitly the obstruction for the reduced equations of a (2+1)-dimensional Yang–Mills Lagrangian with the Chern–Simons term and we find that it is precisely the canonical generator of the second cohomology group of the moduli space of the theory.

The paper is organized as follows. We will review some definitions and geometrical background on higher order tangent bundles in section 2. Section 3 will be devoted to the definition and properties of the variational derivative on 1-forms, the description of self-adjointness and the characterization of locally Lagrangian systems. In section 4 we will discuss a method to decompose closed forms on bundles such that the base space is a strong deformation retract of the total space and we will apply this method to compute the obstructions to the existence of global Lagrangians. Finally we will apply some of the previous results to systems of gauge-invariant differential equations in section 5.

2. Notation and geometric framework

2.1. Higher-order tangent bundles

In this section we will review some basic facts about the geometry of higher-order tangent bundles (we refer the reader to [11, 12] for a detailed account of the subject). Let M be a C^∞-differentiable manifold, we will denote by $\tau_k : T^kM \to M$ the kth order tangent bundle over M, i.e. the set of equivalence classes of curves on M having a contact of kth degree at a given point. There is a natural family of projections $\tau^l_k : T^kM \to T^lM$, $k \geq l$, and it is well known that the maps $\tau^{k-1}_k : T^kM \to T^{k-1}M$ define natural affine bundles, the simplest case being $\tau \equiv \tau^0_1$, the unique that becomes a vector bundle. Local coordinates in T^kM will be denoted by $(q^a, q^a_{(1)}, \ldots, q^a_{(k)})$ where $q^a_{(r)}$ denotes the rth velocity at the point q^a. In these coordinates, $\tau^l_k(q^a, \ldots, q^a_{(k)}) = (q^a, \ldots, q^a_{(l)})$.

Several natural geometric objects are defined on T^kM: the Liouville vector field $\Delta_k \in \mathscr{X}(T^kM)$, a generalization of the dilation vector field on the vector bundle TM, with coordinate expression $\Delta_k = \Sigma^k_{r=0}\, rq_{(r)}\partial/\partial q_{(r)}$; the vertical endomorphism S_k, a $(1, 1)$-tensor field with properties

$$(S_k)^{k+1} = 0 \qquad \mathrm{Im}(S_k)^r = \ker(\tau^{r-1}_k)_* \qquad N_{S_k} = 0$$

where the last equation represents the vanishing of the Nijenhuis tensor field of S_k. In local coordinates $S_k = \Sigma^{k-1}_{r=0}\,(r+1)\partial/\partial q_{(r+1)} \otimes \mathrm{d}q_{(r)}$. There is a canonical immersion $d_{T_k} : T^kM \to T(T^{k-1}M)$ defined by $d_T(q, \ldots, q_{(k)}) = (q, \ldots, q_{(k-1)}; q_{(1)}, \ldots, q_{(k-1)}, q_{(k)})$. The map d_T may also be considered as a vector field along the map τ^{k-1}_k, i.e. $d_T(q, \ldots, q_{(k)}) = \Sigma^{k-1}_{r=0}\, q_{(r+1)}\partial/\partial q_{(r)}$, and therefore defines a map $d_{T_k} : C^\infty(T^{k-1}M) \to C^\infty(T^kM)$, called the total time derivative operator. The operator d_{T_k} defines a first-order differential operator on forms $d_{T_k} : \bigwedge(T^{k-1}M) \to \bigwedge(T^kM)$

satisfying,

$$d_{T_k} \circ d = d \circ d_{T_k} \qquad d_{T_k}(f\alpha) = d_{T_k} f\alpha + f d_{T_k}\alpha$$

(see [13] for a general description on derivations along maps).

Hereafter these three basic objects, the Liouville vector field Δ, the vertical endomorphism S and the total derivative d_T, will be denoted without subscripts, the context showing the actual object we are dealing with. It is easy to see that the action of d_T on forms obeys the well known Cartan identity, $d_T = \mathrm{i}(d_T) \circ d + d \circ \mathrm{i}(d_T)$, where $\mathrm{i}(d_T)$ denotes the contraction with the vector field d_T.

We can also use d_T to define complete lifts of vector fields X on M to T^kM by iteration using the formula $X^{k+1} \circ d_T = d_T \circ X^k$, where X^k denotes the complete lift of X to T^kM. In what follows we will use the same letter X to indicate the different liftings of $X \in \mathscr{X}(M)$ the context indicating again the actual lifting. The advantage of this definition is that it can be used to define liftings of vector fields along arbitrary maps. The set of r-forms along the map $f: M \to N$, i.e. the set of maps $\alpha : M \to \bigwedge^r (T^*N)$ such that $\tau \circ \alpha = f$, will be denoted by $\bigwedge^k (f)$; for instance, $\bigwedge^1 (\tau)$ denotes the set of semibasic forms on TM. An r-form on $\bigwedge^r (\tau_k^{k-l})$ will be called l semibasic. The use of vector fields and forms along maps has been shown to be fruitful in the geometric approach to classical and higher-order mechanics [14].

Commutation relations between d_T, S and $\mathrm{i}(X)$ will be used afterwards. By definition d_T and $\mathrm{i}(X)$ commute; $[S, d_T] = n\mathbb{1}$ on n-forms where S acts on forms as a derivation of zero degree. We can also define $S_{(r)}$ by $S_{(r)}\omega(X_1, \ldots, X_n) = \omega(X_1, \ldots, SX_r, \ldots, X_n)$ and then it is clear that $S\omega = \Sigma_{r=1}^{n} S_{(r)}\omega$. Notice that in general $S_{(r)}\omega$ is not a form. For example, if ω is a 2-form

$$S_{(1)}\omega = \tfrac{1}{2}S\omega + \tfrac{1}{2}g_S$$

where g_S is the symmetric tensor defined by $g_S(X, Y) = \omega(SX, Y) + \omega(SY, X)$.

For any vector field $X \in \mathscr{X}(M)$ and $f \in C^\infty(M)$, the lifting of fX to T^kM is given by the formula

$$(fX)^k = \sum_{r=0}^{k} \frac{1}{r!} d_T^r f S^r X^k \tag{1}$$

that can be proved easily by induction, using the commutation relations between d_T and S.

2.2. *Lagrangian systems*

A natural description of parameter independent ordinary differential equations is carried out on the framework of higher-order tangent bundles by defining a system of lth-order differential equations on M as a submanifold $C \subset T^lM$; locally the submanifold C will be described by a family of functions $F_\alpha(q, q_{(1)}, \ldots, q_{(l)}) = 0$, the local coordinate expression of the system. We will assume that the order of the differential equation is precisely l, in other words, $(\tau_l^{l-1})^{-1}(\tau_l^{l-1}C) \neq C$. Curves γ on M whose lifts to T^lM lie on C are holonomic solutions of the system. A system of differential equations C is in normal form when C is the image of a section σ of τ_l^{l-1}. Systems in normal form define vector fields $\Gamma_\sigma = d_T \circ \sigma$ on $T^{l-1}M$ whose integral curves are lifted curves γ^l from holonomic solutions γ of the system on M.

$$d_T \circ \gamma^{l+1} = \dot{\gamma}^l = \Gamma_\sigma \circ \gamma^l.$$

In what follows we will omit the superscript in the lifting of curves.

Consider now a Lagrangian function L on T^kM and its associated Euler–Lagrange's equations, i.e. the system of differential equations of order $2k$ obtained from Hamilton's variational principle, whose geometric formulation is obtained using the variational derivative on functions $\delta : \bigwedge^1(T^kM) \to \bigwedge^1(\tau_{2k})$ defined by [11, 12]

$$\delta = \sum_{r=0}^{k} \frac{(-1)^r}{r!} d_T^r \circ S^r. \tag{2}$$

The 1-form $\alpha_L = \delta(dL)$ is a 1-form along the map τ_{2k} with local coordinate expression

$$\alpha_L = \sum_{r=0}^{k} (-1)^r \frac{\mathrm{d}^r}{\mathrm{d}t^r} \frac{\partial L}{\partial q^a_{(r)}} \mathrm{d}q^a. \tag{3}$$

Euler–Lagrange's equations associated to L are simply $\alpha_L = 0$, i.e. the submanifold in $T^{2k}M$ defined as the inverse image by α_L of the zero section in T^*M. We will say that a system of ordinary differential equations $\alpha = 0$ in T^lM defined by a 1-form α along the map τ_l is locally Lagrangian if there exists a function L locally defined in T^lM such that $\alpha = \delta(\mathrm{d}L)$.

In terms of the Poincaré–Cartan form

$$\theta_L = \sum_{r=0}^{k-1} \frac{(-1)^r}{(r+1)!} d_T^r \circ S^{r+1}(\mathrm{d}L) \tag{4}$$

the Euler–Lagrange equations (3) take the form $\alpha_L = \mathrm{d}L - d_T\theta_L = 0$. For regular Lagrangian functions, the Euler–Lagrange equations are in normal form, i.e. there exists a unique section σ_L of τ_{2k}^{2k-1} such that $\alpha_L \circ \sigma_L = 0$; the associated vector field $\Gamma_L = d_T \circ \sigma_L$ is the unique solution of the dynamical equation $\mathrm{d}L - \mathscr{L}_{\Gamma_L}\theta_L = 0$, or equivalently, $\mathrm{i}(\Gamma_L)\omega_L = \mathrm{d}E_L$ with $E_L = \mathrm{i}(d_T)\theta_L - L$, and $\omega_L = -\mathrm{d}\theta_L$ the exact symplectic form associated to L.

3. The variational derivative and locally Lagrangian systems

3.1. *The variational derivative on 1-forms*

Let $\mathscr{C}(M)$ be the set of differentiable curves γ on the manifold M. Given a Lagrangian $L \in C^\infty(T^kM)$, the action integral is the local functional $\mathscr{S}$ on $\mathscr{C}(M)$

$$\mathscr{S}(\gamma) = \int L(\gamma(t))\, \mathrm{d}t.$$

Hamilton's variational principle is formulated on the submanifold of curves with fixed endpoints x_1, $x_2 \in T^{k-1}M$, $\mathscr{C}_{1,2}(M) = \{\gamma : [t_1, t_2] \to M \mid \gamma^{k-1}(t_i) = x_i \in T^{k-1}M\}$, and asserts that the possible evolution curves associated to the Lagrangian L are given by the critical points of $\mathscr{S}$ on $\mathscr{C}_{1,2}(M)$, i.e., solutions of the equation

$$\mathrm{d}\mathscr{S}(\gamma) = 0 \qquad \gamma \in \mathscr{C}_{1,2}(M). \tag{5}$$

A tangent vector to the set $\mathscr{C}_{1,2}$ at the curve γ, $X \in T_\gamma\mathscr{C}_{1,2}(M)$, is an equivalence class of curves $\rho(s) = \gamma_s$ on $\mathscr{C}_{1,2}(M)$ agreeing at first order on $\gamma = \gamma_0$, i.e. a class of maps $\rho : \mathbb{R} \times I \to M$ defining a vector field $X : I \to TM$ along γ, $\tau \circ X = \gamma$, vanishing at the endpoints, $X(t_i) = 0$ (note that vanishing at endpoints extends to liftings of X until $T^{k-1}M$, and then higher-order liftings are vertical with respect to τ_l^{k-1}, $l \geqslant k$, at the endpoints). The differential of $\mathscr{S}$ at γ evaluated on X can be expressed as

$$\langle \mathrm{d}\mathscr{S}(\gamma), X \rangle = \int_I \gamma^*(\mathrm{i}(X)\, \mathrm{d}L) = \int_I \langle \mathrm{d}L(\gamma(t)), X(t) \rangle\, \mathrm{d}t.$$

The symbol X on the right-hand side stands for an extension to an open neighbourhood in M containing $\gamma(I)$ of the vector field X along the curve γ. If we try to find solutions of Hamilton's variational principle contracting equation (5) with an arbitrary tangent vector $X \in T_\gamma \mathscr{C}_{1,2}(M)$ we cannot impose directly the vanishing of the integrand; integration by parts must be carried out, factorizing the arbitrary variation X and obtaining Euler–Lagrange's equations. Integration by parts can be implemented geometrically multiplying X by an arbitrary function $f \in C^\infty(M)$, the new vector X_f on $T_\gamma \mathscr{C}_{1,2}(M)$ is $X_f(t) = f(\gamma(t))X(t)$ and we have

$$\begin{aligned}
\langle \mathrm{d}\mathscr{S}, X_f\rangle(\gamma) &= \int_I \gamma^*(\mathrm{i}((fX)^k)\,\mathrm{d}L) \\
&= \int_I \gamma^*\left(\sum_{r=0}^{k} \frac{1}{r!} d_T^r f \mathrm{i}(S^r X)\,\mathrm{d}L\right) \\
&= \sum_{r=0}^{k} \int_I \frac{\mathrm{d}^r f}{\mathrm{d}t^r} \gamma^*\left(\frac{1}{r!}\mathrm{i}(X) S^r(\mathrm{d}L)\right) \\
&= \sum_{r=0}^{k} \int_I f\gamma^*\left(\frac{(-1)^r}{r!}\mathrm{i}(X) d_T^r \circ S^r(\mathrm{d}L)\right) = 0
\end{aligned}$$

that must vanish for every f, then the factor in the last parenthesis defines a system of differential equations, the Euler–Lagrange equations $\alpha_L = 0$ presented before. In the computation above we have used the lifting formula (1), commutation of d_T with $\mathrm{i}(X)$, boundary conditions for X and ordinary integration by parts. The variational derivative δ defined in equation (2) emerges from the last expression, and property $S \circ \delta = 0$ is obtained directly noticing the dependence of the integrand on f and not on its derivatives. Notice that even if δ is said to act on functions, in fact, it is a first-order differential operator on the set of 1-forms in T^kM.

We shall consider now a local 1-form Φ on the manifold $\mathscr{C}(M)$ defined by

$$\langle \Phi, X\rangle(\gamma) = \int_I \gamma^*\langle \phi, X\rangle \qquad X \in T_\gamma \mathscr{C}(M)$$

where ϕ is a 1-form in $\bigwedge^1(T^kM)$. Again the exterior differential $\mathrm{d}\Phi$ is easily computed in terms of the 1-form ϕ, but as in the case of functionals, closedness of Φ is not equivalent to closedness of ϕ. Integration by parts must be carried on and a similar computation will give

$$\mathrm{d}\Phi(X_f, Y)(\gamma) = \sum_{r=0}^{k} \int_I f\gamma^*\left(\mathrm{i}(Y)\mathrm{i}(X)\frac{(-1)^r}{r!} d_T^r \circ S_{(1)}^r\,\mathrm{d}\phi\right).$$

Because of the operator in the parenthesis above, a variational derivative on 1-forms can be defined as the first-order differential operator acting on 2-forms $\delta_1 : \bigwedge^2(T^kM) \to \mathscr{T}^2(T^{2k}M)$ given by the formula

$$\delta_1 = \sum_{r=0}^{k} \frac{(-1)^r}{r!} d_T^r \circ S_{(1)}^r \tag{6}$$

where $\mathscr{T}^2$ denotes the bundle of 2-covariant tensors. Dependence on f and not on its derivatives shows that $S_{(1)} \circ \delta_1 = 0$. An alternative variational derivative $\delta_2 = \Sigma_{r=0}^k (-1)^r/r!\, d_T^r \circ S_{(2)}^r$ is obtained considering the modified vector field fY on the second argument. Because $\delta_1(\mathrm{d}\phi)$ is not antisymmetric, we can decompose δ_1 into its symmetric δ^S, and antisymmetric part δ^A,

$$\delta^A = \frac{1}{2}(\delta_1 + \delta_2) = \mathbb{1} + \frac{1}{2}\sum_{r=1}^{k} \frac{(-1)^r}{r!} d_T^r \circ [S^r] \tag{7}$$

$$\delta^S = \frac{1}{2}(\delta_1 - \delta_2) = \frac{1}{2}\sum_{r=1}^{k} \frac{(-1)^r}{r!} d_T^r \circ g_{S^r} \tag{8}$$

with $g_{S^r}(X, Y) = \omega(S^r X, Y) - \omega(X, S^r Y)$ and $[S^r]\omega(X, Y) = \omega(S^r X, Y) + \omega(X, S^r Y) \neq S(\overset{r-2}{\cdots}(S\omega)\ldots)$.

If ϕ is a semibasic 1-form, for instance a 1-form α representing a system of Euler–Lagrange differential equations, the 2-form $\mathrm{d}\alpha$ vanishes over two vertical vectors and S^r can be written instead of $[S^r]$ on the expression of δ^A in equation (7). It is also easy to show from (8) that the symmetric variational derivative $\delta^S(\mathrm{d}\alpha)$ is a total time derivative $\delta^S(\mathrm{d}\alpha) = d_T\beta$, therefore it is not relevant in the computation of $\mathrm{d}\Phi$ because integration of $d_T\beta(X, Y)$ yields the values of $\beta(X, Y)$ at the endpoints that vanish because of the boundary conditions. The previous discussion is summarized in the following proposition.

Proposition 1. Let Φ be a local 1-form on $\mathscr{C}_{1,2}(M)$ defined by the 1-form ϕ on T^kM by $\Phi(\gamma) = \int \gamma^*\phi$. then, $\mathrm{d}\Phi = \int \gamma^*(\delta^A(\mathrm{d}\phi))$ and in consequence Φ is closed iff $\delta^A(\mathrm{d}\phi) = 0$.

The variational derivative can be defined in a similar way for forms of higher degree, but such a generalization is not necessary here. In what follows we will use the notation δ to indicate either the antisymmetric variational derivative on 1-forms δ^A, equation (7), or the ordinary variational derivative on functions δ defined by the equation (2). A 1-form ϕ such that $\delta(\mathrm{d}\phi) = 0$ will be said to be δ-closed and notice that in general $\delta^2 \neq 0$ but, $\delta d\delta d = 0$.

3.2. *δ-closed and self-adjoint differential equations*

As we have already pointed it out in the introduction, it is well known that a system of partial differential equations is localy Lagrangian iff the local Euler–Lagrange 1-form ξ defined by the equations in the space of fields is closed. We could use the description of the exterior differential in the space of curves given in proposition 1 to restate this condition in terms of the geometry of T^kM for systems of ordinary differential equations.

Theorem 1. For every Lagrangian function $L \in C^\infty(T^kM)$, its associated Euler–Lagrange equations $\alpha_L = \delta(\mathrm{d}L)$ are δ-closed. Conversely, a system of differential equations $\alpha = 0$ defined by a 1-form $\alpha : T^lM \to T^*M$ along the map τ_l such that α is δ-closed is locally Lagrangian.

Proof. The Euler–Lagrange 1-form ξ_L on the space of curves defined by the Lagrangian L is given by $\xi_L(\gamma) = \mathrm{d}\mathscr{S}(g) = \int \gamma^*(\delta(\mathrm{d}L)) = \int \gamma^*\alpha_L$. Then $\mathrm{d}\xi_L = 0$ implies automatically $\delta(\mathrm{d}\alpha_L) = 0$ (proposition 1).

On the other hand, given $\alpha \in \bigwedge^1(\tau_l)$ such that $\delta(\mathrm{d}\alpha)=0$ it is easy to show that the 2-form ω given by

$$\omega=-\frac{1}{2}\sum_{r=1}^{l}\frac{(-1)^r}{r!}\,d_T^{r-1}S^r(\mathrm{d}\alpha) \tag{9}$$

satisfies $\mathrm{d}\alpha=d_T\omega$. Let us note first that the kernel of the operator d_T is trivial on forms of degree $\geqslant 1$ and consists of constant functions on forms of zero degree. It means that if $\phi=d_T\beta$ and ϕ is on T^lM then β is on $T^{l-1}M$; moreover, if $\mathrm{d}\phi=d_T\beta$ then β has to be closed because $0=\mathrm{d}\phi=d_T\,\mathrm{d}\beta$. Therefore from $\mathrm{d}\alpha=d_T\omega$ it follows that $\omega\in\bigwedge^2(T^{l-1}M)$ and $\mathrm{d}\omega=0$. Let θ be a locally defined 1-form such that $\omega=-\mathrm{d}\theta$; the 1-form $\rho=\alpha+d_T\theta$ is closed and locally there exists a function L on T^lM such that $\rho=\mathrm{d}L$. It is easy to check that

$$\theta_L=\sum_{r=0}^{l-1}\frac{(-1)^r}{(r+1)!}\,d_T^r\circ S^{r+1}(\rho)=\theta$$

and finally $\alpha=\mathrm{d}L-d_T\theta_L=\alpha_L$. □

It happens that if the 1-form α is in $\bigwedge^1(\tau_{2k})$ the local Lagrangian function can always be chosen to be on T^kM; other equivalent Lagrangians with dependence in higher-order velocities are obtained by adding total time derivatives. The reason for that is that the 2-form ω defined by equation (9) always admits a local expression $\omega=\mathrm{d}\theta$ with θ a k-semibasic form as shown in the following proposition.

Proposition 2. Let $\alpha\in\bigwedge^1(\tau_{2k})$ be δ-closed, then there always exists a local Lagrangian L for α on T^kM.

Proof. From the definition of ω in equation (9), and the properties of S^r and d_T^r, it follows that

$$\omega\,\big|_{\ker(\tau_{2k-1}^{k-1})_*}=0$$

i.e. ω vanishes over two vertical vectors along the map τ_{2k-1}^{k-1}. Then the locally defined 1-form θ such that $\mathrm{d}\theta=\omega$ can be chosen to be in $\bigwedge^1(\tau_{2k-1}^{k-1})$, and the Lagrangian form $\rho=\alpha+d_T\theta$ results to be projectable to T^kM. In fact, computing the variation of ρ along vector fields in $\ker(\tau_{2k}^k)_*$ we get

$$\mathscr{L}_V\rho=\mathrm{d}(\mathrm{i}(V)d_T\theta)\qquad\forall V\in\ker(\tau_{2k}^k)_*$$

but $d_T\theta\in\bigwedge^1(\tau_{2k}^k)$ and then $\mathscr{L}_V\rho=0$, $\forall V\in\ker(\tau_{2k}^k)_*$, hence $\rho=\mathrm{d}L\in\bigwedge^1(T^kM)$. □

We will close this section discussing self-adjointness of a system of differential equations from the geometrical point of view described before. Consider a vector field X in M and a 1-form $\alpha: T^lM\to T^*M$ defining a system of differential equations $C=\alpha^{-1}(0)$. The variation $\mathscr{L}_X\alpha$ of α under X will be denoted by α_X and the invariance of S under X implies that α_X is again a 1-form over τ_l. The adjoint variation of α under X, α_X^*, is defined by $\alpha_X^*=\delta(\mathrm{d}(\mathrm{i}(X)\alpha))$. Taking into account that $\mathrm{i}(X)\alpha\in C^\infty(T^lM)$ we find $\alpha_X^*\in\bigwedge^1(\tau_{2l})$. The Cartan identity for the Lie derivative and $S(\mathscr{L}_X\alpha)=0$ shows us that

$$\alpha_X^*=\alpha_X-\delta(\mathrm{i}(X)\,\mathrm{d}\alpha).$$

A 1-form α is self-adjoint if $\alpha_X = \alpha^*_X\ \forall X \in \mathscr{X}(M)$, i.e. if $\delta(\mathrm{i}(X)\,\mathrm{d}\alpha) = 0, \forall X \in \mathscr{X}(M)$ (notice that the local coordinate expression of self-adjointness associated with our geometric definition coincides with that of [25], although the definition of variation and adjoint variation does not). The condition of self-adjointness on α is now easily shown to be equivalent to δ-closedness because

$$\delta(\mathrm{i}(X)\,\mathrm{d}\alpha) = \sum_{r=0}^{k} \frac{(-1)^r}{r!} d^r_T S^r(\mathrm{i}(X)\,\mathrm{d}\alpha) = \mathrm{i}(X) \sum_{r=0}^{k} \frac{(-1)^r}{r!} d^r_T S^r_{(2)}\,\mathrm{d}\alpha = \mathrm{i}(X)\delta_2\,\mathrm{d}\alpha$$

but the space of lifted vector fields is an ample space for the tangent bundle $T(T^kM)$ then $\delta(\mathrm{i}(X)\,\mathrm{d}\alpha) = 0$ for all X is equivalent to $\delta_2(\mathrm{d}\alpha) = 0$, hence equivalent to $\delta(\mathrm{d}\alpha) = 0$.

In the particular case of SODEs, the condition of self-adjointness becomes

$$\mathrm{d}\alpha = \tfrac{1}{2} d_T S(\mathrm{d}\alpha) - \tfrac{1}{4} d^2_T S^2(\mathrm{d}\alpha).$$

This is precisely the intrinsic definition of the necessary and sufficient condition for the system $\alpha = 0$ to be locally Lagrangian. In fact, last condition is equivalent to the separate equations

$$\mathrm{d}\alpha = \tfrac{1}{2} d_T S(\mathrm{d}\alpha) \qquad S^2(\mathrm{d}\alpha) = 0$$

as can be seen using that α is semibasic.

4. Obstructions to the existence of global Lagrangians

In this section we will consider the problem of deciding whether there exists a global Lagrangian for a locally Lagrangian system of equations or not. For this purpose we will prove the following decomposition theorem.

4.1. *Decomposing forms on bundles*

Let $E \xrightarrow{\pi} M$ be a locally trivial fibre bundle having global sections and such that there is a strong deformation retraction ϕ_t of E into M along the fibres. We will use an adaptation of the proof of the relative Poincaré theorem to decompose a closed r-form ω on the bundle $E \to M$ into the sum of an exact form on E and a closed form on M.

Let ω be an r-form on the manifold N and X_t a time-dependent vector field on N, i.e. a family of maps $X_t : N \to TN$, such that $X_t(m)$ is tangent to $\phi_t(m)$ for all $m \in N$ [16]. The variation of the form ω along the time-dpendent vector field X_t is given by the Lie derivative formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi^*_t\omega = \phi^*_t(\mathrm{i}(X_t)\,\mathrm{d}\omega) + \mathrm{d}\phi^*_t(\mathrm{i}(X_t)\omega). \tag{10}$$

Choosing an open set on N such that the integral curves of X_t contain the interval [0, 1], integrating (10) and denoting by I the map sending ω into $\int_0^1 \phi^*_t(\mathrm{i}(X_t)\omega)\,\mathrm{d}t$, we have for a closed form ω

$$\phi^*_1\omega - \phi^*_0\omega = \mathrm{d}(I\omega). \tag{11}$$

Consider now the particular case of N being the total space of a bundle $E \xrightarrow{\pi} M$ with the properties stated above, and let the time-dependent vector field X_t be the time-dependent vector field Δ_t tangent to the curves $\phi_t(\xi)$, $\forall \xi \in E$, defined by the strong deformation retraction ϕ_t; formula (11) becomes

$$\omega = \mathrm{d}(I\omega) + \phi^*_0\omega \tag{12}$$

the map ϕ_1 is the identity on E and ϕ_0 is precisely the projection map π identifying M with a submanifold of E defined by some global section. It is clear that $\phi_0^*\omega$ is a basic form on E and therefore there exists a closed form $P\omega$ on M such that $\pi^*(P\omega)=\phi_0^*\omega$. We will call $P\omega$ the projectable part of ω, and P induces an isomorphism between the de Rham cohomology rings $H^*(E)$ and $H^*(M)$. Notice that if the closed form ω belongs to $\bigwedge_1^r(E)$, i.e., ω vanishes over two vertical fields on E, then $I\omega$ is semibasic. Namely, for any vertical vector field V

$$\mathrm{i}(V)I\omega=\int_0^1 \mathrm{i}(V)\phi_t^*(\mathrm{i}(\Delta_t)\omega)\,\mathrm{d}t=\int_0^1 \phi_t^*(\mathrm{i}((\phi_{-t})_*V)\mathrm{i}(\Delta_t)\omega)\,\mathrm{d}t=0$$

because $(\phi_{-t})_*V$ is a vertical field for any t. Then we have the following proposition.

Proposition 3. Let $E\xrightarrow{\pi}M$ be a bundle satsifying the conditions stated in the previous paragraph, and ω a closed r-form in $\bigwedge_1^r(E)$, then ω decomposes as

$$\omega=\mathrm{d}\theta+\pi^*P\omega \tag{13}$$

where $\theta=I\omega$ is a semibasic form on E and $P\omega$ is a closed r-form on M and this decomposition is unique up to the addition of exact forms in M.

Let us consider now projections of higher-order tangent bundles $\tau_l^k: T^lM\to T^kM$. It is clear that the projections τ_l^k have global sections, and pasting together using a partition of unity, the locally defined family of maps $\phi_t(q, q_{(1)},\ldots,q_{(l)})=(q, q_{(1)},\ldots,q_{(k)}, t^{k+1}q_{(k+1)},\ldots,t^l q_{(l)})$, results that T^kM is a strong deformation retraction of T^lM. Denoting by $(\Delta_l^k)_t$ the time-dependent vector field tangent along the curves ϕ_t, we have the following corollary.

Corollary 1. Let ω be a closed r-form on T^lM which vanishes when applied to two vertical vector fields along the map τ_l^k, i.e. $\omega\in\bigwedge_1^r(E)$, with $E=T^lM$ and $\pi=\tau_l^k$. Then there exists an essentially unique $(l-k)$-semibasic $(r-1)$-form θ on T^lM and a closed r-form $P\omega$ on T^kM related by the formula

$$\omega=\mathrm{d}\theta+(\tau_l^k)^*(P\omega). \tag{14}$$

4.2. *Obstructions*

If we have a system of δ-closed differential equations $\alpha\in\bigwedge^1(\tau_{2k})$, i.e. a locally Lagrangian system, then the 2-form $\omega\in\bigwedge^2(T^{2k-1}M)$ associated to α by equation (9), is closed and lies in $\bigwedge_1^2(E)$, $E=T^{2k-1}M$ and $\pi=\tau_{2k-1}^{k-1}$. Applying the decomposition formula (14) to ω with $l=2k-1$, we get

$$\omega=\mathrm{d}\theta+(\tau_{2k-1}^{k-1})^*P\omega \tag{15}$$

where θ is a k-semibasic 1-form on $T^{2k-1}M$ and $P\omega$ is a closed 2-form on $T^{k-1}M$. Denoting $P\omega$ by ω_k and applying again the decomposition theorem (14) to ω_k with $l=k-1$ and $k=0$ we get

$$\omega_k=\mathrm{d}\psi+\tau_{(k-1)}^*(P\omega_k) \tag{16}$$

where ψ is a 1-form on $T^{k-1}M$, not necessarily semibasic, and $P\omega_k$ is a closed 2-form on M denoted in what follows by Ω. Notice that if α is a SODE, $k=1$, the second decomposition, equation (16) is irrelevant and $P\omega$ is already a closed 2-form in M.

The cohomology class $[\Omega] \in H^2(M)$ of Ω will be called the (first) obstruction to the existence of a global Lagrangian for the system $\alpha = 0$. If Ω is exact we will say that there are no essential obstructions to the existence of a global Lagrangian for α. In general we will say that the (first) obstruction $[\Omega]$ can be removed if the 2-form Ω is of integer class. In that case, it is well known that we can replace M by a circle bundle $P \xrightarrow{p} M$ and there is a connection 1-form A on P such that $p^*\Omega = \mathrm{d}A$. Then pulling back the decomposition of ω, equation (15), and ω_k, equation (16), to $T^{2k-1}P$ we find that $p^*\omega$ is exact

$$p^*\omega = \mathrm{d}[p^*\theta + (\tau_{2k-1}^{k-1} \circ p)^*\psi + \tau_{2k-1}^* A] = \mathrm{d}\theta_A. \tag{17}$$

Assuming now that the obstruction $[\Omega]$ can be removed, we will denote $p^*\omega$, $p^*\theta$, $p^*\psi$ and p^*A simply by ω, θ, ψ and A respectively; the circle bundle P will be denoted again by M and we will consider the k-semibasic 1-form θ_A on $T^{2k-1}M$. In theorem 1 we proved that if α is locally Lagrangian the Lagrangian 1-form $\rho = \alpha - d_T\theta_A$ is closed. Applying once more the decomposition theorem to ρ with $l = k$ and $k = 0$, formula (14), we have

$$\rho = \mathrm{d}(I\rho) + \tau_k^*(P\rho) \tag{18}$$

and denoting the 0-form $I\rho$ by L_c we have

$$L_c = \int_0^1 \phi_t^*(\mathrm{i}(\Delta_k)\rho)\,\mathrm{d}t \tag{19}$$

where Δ_k is the Liouville vector field described in subsection 2.1. We call the cohomology class of the closed 1-form $P\rho$ the second obstruction to the existence of a global Lagrangian for α. This obstruction is very mild because we can always replace the configuration space M by its universal covering $\tilde{M}$ and then $H^1(\tilde{M}) = 0$. In $\tilde{M}$ we will have $P\rho = -\mathrm{d}V$ for some (potential) function V, and formula (18) becomes

$$\rho = \mathrm{d}(L_c - \tau_k^* V) \tag{20}$$

and the global Lagrangian on T^kM for the system α is $L = L_c - \tau_k^* V$. It is remarkable that all terms appearing on equation (20) are explicitly computed from α. In the particular case of SODEs, $k = 1$, the global Lagrangian obtained in this way has a natural affine approximation L_a given by the affine function along the fibres of TM

$$L_a = \hat{A} - \pi^* V \tag{21}$$

where $\hat{A}$ denotes the evaluation of the connection 1-form A on the vectors of TM, i.e. $\hat{A}(v) = A_{\tau(v)}(v)$, for all $v \in TM$. The non-affine part of L_c will be denoted by L_{nl} and consequently we have

$$L = L_{nl} + \hat{A} - \pi^* V. \tag{22}$$

The decomposition (22) of the global Lagrangian L of SODEs gives at the same time the local normal form of locally Lagrangians systems; L_{nl} is the nonlinear term in velocities, $\hat{A}$ the minimal coupling term, and V the potential function.

5. Applications and examples

5.1. Gauge invariant systems

Dynamical systems obtained as reduction of gauge-invariant systems play an important role in physical theories. Consider the system of differential equations $\alpha = 0$ defined by the 1-form $\alpha \in \bigwedge^1(\tau_l)$ and suppose that G is a group of symmetries of α, i.e. a group of transformations of M such that the lifted action of G to T^lM preserves α, i.e. $\mathscr{L}_{X_a}\alpha = 0$, $\forall a \in \mathfrak{g}$. There is a natural action of the lth tangent group T^lG on T^lM defined by the formula $j^l(g) \cdot j^l(\gamma) = j^l(g \cdot \gamma)$ where $j^l(g)$ and $j^l(\gamma)$ represent the equivalence class containing the lth lifting of the curves g and γ in G and M respectively. The system of equations $\alpha = 0$ is gauge-invariant with respect to the group G if α is invariant with respect to the action of T^lG in T^lM; in particular a gauge-invariant system with respect to the group G possess G as a group of symmetries (see [17] for a motivation of this definition and [18] for a discussion on cohomological aspects of the equivariant inverse problem in the particular case of SODEs). If α is gauge invariant with respect to G, α projects to the quotient space $T^lM/T^lG \simeq T^l(M/G)$ defining a 1-form α_G in $T^l(M/G)$ along the map $\tau_l^G : T^l(M/G) \to M/G$, $\alpha_G \in \bigwedge^1(\tau_G^l)$. The system of differential equations $\alpha_G = 0$ will be called the reduced system of α by G.

We will assume that M/G is a manifold and the canonical projection $\pi_G : M \to M/G$ is a submersion. It is clear from figure 1 below that if we denote by $\pi_G^l : T^lM \to T^l(M/G)$ the canonical projection map along π_G we have $\pi_G^* \circ \alpha_G \circ \pi_G^l = \alpha$. The variational derivatives on 1-forms, corresponding to the exterior differentials on $\mathscr{C}_{1,2}(M)$ and $\mathscr{C}_{1,2}(M/G)$ respectively, δ and δ_G, are related by $(\pi_G^{2l})^* \circ \delta_G = \delta \circ (\pi_G^l)^*$ (see figure 2; notice that $(\pi_G^l)^*$ acts in forms as $(\pi_G^l)^*\alpha_G = \pi_G^* \circ \alpha_G \circ \pi_G^l$).

$$\begin{tikzcd} T^lM \arrow[r, "\alpha"] \arrow[d, "\pi_G^l"'] & T^*M \\ T^l(M/G) \arrow[r, "\alpha_G"'] & T^*(M/G) \arrow[u, "\pi_G^*"'] \end{tikzcd}$$

Figure 1

$$\begin{tikzcd} \bigwedge^2(T^lM) \arrow[r, "\delta"] & \bigwedge^2(\tau_{2l}) \\ \bigwedge^2(T^l(M/G)) \arrow[r, "\delta_G"'] \arrow[u, "(\pi_G^l)^*"] & \bigwedge^2(\tau_{2l}^G) \arrow[u, "(\pi_G^{2l})^*"'] \end{tikzcd}$$

Figure 2

Proposition 4. Let $\alpha \in \bigwedge^1(\tau_l)$ be a gauge-invariant system of differential equations with respect to the group G, then α is locally Lagrangian iff the reduced system α_G is locally Lagrangian.

Proof. All we have to show using theorem 1 is that α_G is δ_G-closed iff α is δ-closed, but this is clear because

$$\begin{aligned}(\pi_G^{2l})^*\delta_G(\mathrm{d}\alpha_G) &= \delta(\pi_G^l)^*(\mathrm{d}(\alpha_G)) = \delta(\mathrm{d}(\pi_G^l)^*\alpha_G) \\ &= \delta(\mathrm{d}\pi_G^* \circ \alpha_G \circ \pi_G^l) = \delta(\mathrm{d}\alpha) = 0. \qquad \square\end{aligned}$$

If the reduced system α_G is locally Lagrangian, we can try to relate the obstruction Ω_G to the existence of a global Lagrangian for it and the obstruction Ω for the unreduced system α. It is obvious that the natural lifting of the action of G to T^lM preserves the vector field defined by the retraction of T^lM to T^kM and in consequence the decomposition formulae (14), (15) and (16) are equivariant, i.e. all the terms appearing in it ω, θ and $P\omega$ are invariant under G, and then the obstruction form Ω is projectable to M/G. This proves the following theorem.

Theorem 2. (Equivariant obstruction theorem). Under the conditions stated in the paragraphs above, the cohomology class of the form Ω_G in M/G determining the existence of a global Lagrangian for the reduced system α_G is precisely the cohomology class of the projection of the obstruction Ω of α.

This result is particularly interesting when we do not really want to compute the quotient space M/G. It allows us to identify the reduced obstruction Ω_G in terms of the geometry of the unreduced space M. It is simple to see that even if the unreduced system is globally Lagrangian, the reduced system does not have to be so because the cohomology class of Ω can vanish on M but at the same time induce a non-trivial cohomology class in the quotient space. For instance, if Ω is exact and G does not preserve any potential 1-form of Ω, then the induced obstruction would not be zero in general and the reduced system will not have to be globally Lagrangian. If the group G is compact and Ω is exact, there always exists a G-invariant potential for it and the reduced system is Lagrangian.

As we indicated before, systems of interest in physical theories are usually gauge invariant systems of SODEs $\alpha = 0$ in TM. In this particular situation the reduced system α_G is defined in $T(M/G)$ and as we have seen before the obstruction form Ω projects to Ω_G. For example, consider the equations describing the motion of a charged particle in the magnetic field created by a monopole (see [10] for a thorough discussion of this system). It happens that this SODE is locally Lagrangian but not globally Lagrangian. A simple computation shows the obstruction to the existence of a global Lagrangian is precisely the strength of the monopole's magnetic field. This obstruction can be removed iff the Dirac quantization condition for the electric charge and the monopole's magnetic charge holds, i.e. if the obstruction is of integer class. Removing the obstruction amounts to the substitution of the original configuration space of the particle by a circle bundle over it. The equations of motion in the total space become globally Lagrangian and gauge invariant with respect to the natural action of U(1). The equivariant obstruction theorem can be applied and after projection, we recover the initial equations and the original obstruction. In the following section we will discuss an interesting infinite-dimensional analogue of the monopole-electron system that as in the previous discussion is globally Lagrangian before reduction and becomes locally Lagrangian after reduction unless some quantization condition holds.

5.2. (2+1)-dimensional Yang–Mills equations with Chern–Simons term

We shall consider the dynamical system obtained from a (2+1)-dimensional Yang–Mills theory with the Chern–Simons term. Let $P(G, M)$ be a principal fibre bundle with stuctural group G, a compact connected simple Lie group, for example SU(2), over a three-dimensional Riemannian manifold (M, η), such that $M \cong \Sigma \times \mathbb{R}$ for some Riemann surface Σ. The configuration space of the system is $\mathscr{A} \times \mathscr{A}_0$ where $\mathscr{A}$ denotes the affine space of irreducible connections on $P(G, \Sigma)$ and $\mathscr{A}_0$ is the space of section of the adjoint bundle ad P over Σ, elements of $\mathscr{A}_0$ are locally functions $\Sigma \to \mathfrak{g}$. The Lagrangian on $T(\mathscr{A} \times \mathscr{A}_0)$ is given by [19]

$$L(A, \dot{A}; A_0, \dot{A}_0) = \frac{1}{2} \|\dot{A} - d_A A_0\|^2 - \frac{1}{2} \|F(A)\|^2 + \frac{\lambda}{4\pi} (2\langle A_0, *F\rangle + \langle A, *\dot{A}\rangle) \qquad (23)$$

where $F(A)$ denotes the curvature of the connection A, $\|\cdot\|$ denotes the L^2 norm on the space of ad P-valued forms over Σ associated to the product

$$\langle \alpha, \beta \rangle = \int_\Sigma \operatorname{Tr}(\alpha \wedge *\beta) \qquad \forall \alpha, \beta \in \Omega^k(\Sigma, \operatorname{ad} P)$$

and $*$ is the Hodge operator defined by any Riemannian metric on the conformal class of the complex structure of the Riemann surface Σ. Denoting by $\mathscr{G}$ the group of gauge transformations of $P(G, \Sigma)$ acting on the configuration space as

$$g \cdot (A, A_0) = (gAg^{-1} + g\, \mathrm{d}g^{-1}, gA_0 g^{-1}) \qquad g \in \mathscr{G}, (A, A_0) \in \mathscr{A} \times \mathscr{A}_0 \tag{24}$$

the vector field associated to any infinitesimal generator of the group $\mathscr{G}$, $\xi \in L(\mathscr{G}) = \Omega^0(\Sigma, \operatorname{ad} P)$ is given by $X_\xi(A, A_0) = (d_A \xi, [A_0, \xi])$, where d_A denotes the covariant differential $d_A = d + [A, \cdot\,]$. The complete and vertical liftings of this vector field to $T(\mathscr{A} \times \mathscr{A}_0)$ are given by

$$X_\xi^c(A, \dot A; A_0, \dot A_0) = (d_A \xi, [\dot A, \xi]; [A_0, \xi], [\dot A_0, \xi]) \tag{25}$$

$$X_\xi^V(A, \dot A; A_0, \dot A_0) = (0, d_A \xi; 0, [A_0, \xi]). \tag{26}$$

It is apparent that L is not invariant under the action of the complete lifting (25) of the action of the group $\mathscr{G}$ but it is gauge invariant. The true phase space of the system is obtained quotienting out the action of the group of gauge transformations. Usually this is done using Dirac's theory of constraints in the canonical formalism to get rid of the constraints introduced by the degenerate Lagrangian L. It is also possible to reduce directly the Lagrangian system $(T(\mathscr{A} \times \mathscr{A}_0), L)$ as indicated in the previous section or following the discussion in [13]. We are just interested in computing the obstruction for the existence of a global Lagrangian on the quotient space $T(\mathscr{A}/\mathscr{G})$. Then we will first compute the Poincaré-Cartan 1-form θ_L defined by the Lagrangian (23) obtaining

$$\theta_L = \left(\dot A - d_A A_0 + \frac{\lambda}{4\pi} * A \right) \delta A. \tag{27}$$

The Yang-Mills Cartan form ω_L is obtained computing the exterior differential $\mathrm{d}\theta_L$ and we get

$$\omega_L = \delta \dot A \wedge \delta A + 2[A_0, \delta A] \wedge \delta A + d_A^* \delta A \wedge \delta A_0 + \frac{\lambda}{4\pi} * \delta A \wedge \delta A. \tag{28}$$

Notice that the kernel of ω_L is spanned by the vector fields of the form

$$X(A, \dot A; A_0, \dot A_0) = \Phi \frac{\delta}{\delta A_0} + d_A \Phi \frac{\delta}{\delta \dot A} + \Psi \frac{\delta}{\delta \dot A_0}$$

with arbitrary $\Phi, \Psi \in \Omega^0(\Sigma, \operatorname{ad} P)$. In other words $\ker \omega_L = T\mathscr{A}_0 \oplus L\mathscr{G}^V$ and a simple computation shows that ω_L is invariant under the action of the Lie group $T\mathscr{G}$. We are in consequence in the situation of the equivariant decomposition theorem and we can compute the obstruction to the existence of a global Lagrangian in the reduced tangent bundle $T(\mathscr{A}/\mathscr{G})$ by just looking at the decomposition of ω_L and then projecting the obstruction form Ω. The obstruction Ω is given by $\Omega = (\lambda/4\pi) * \delta A \wedge \delta A$, or more explicitly

$$\Omega(\alpha, \beta) = -\frac{\lambda}{2\pi} \int_\Sigma \operatorname{Tr}(\alpha \wedge \beta) \qquad \forall \alpha, \beta \in T_A(\mathscr{A}). \tag{29}$$

It is well known that Ω is a $\mathcal{G}$-invariant symplectic form on $\mathcal{A}$ inducing the generator of the second cohomology group of the moduli space $\mathcal{M} = \mathcal{A}/\mathcal{G}$ and is of integer class iff λ is integer. When this quantization condition is satisfied, the obstruction can be removed and a global Lagrangian exists on the tangent space of a circle bundle over $\mathcal{M}$.

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